BOUNDARY CONDITIONS ASSOCIATED WITH THE PAINLEVÉ III$'$ AND V EVALUATIONS OF SOME RANDOM MATRIX AVERAGES

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Abstract. In a previous work a random matrix average for the Laguerre unitary ensemble, generalising the generating function for the probability that an interval $(0,s)$ at the hard edge contains k eigenvalues, was evaluated in terms of a Painlevé V transcendent in σ-form. However the boundary conditions for the corresponding differential equation were not specified for the full parameter space. Here this task is accomplished in general, and the obtained functional form is compared against the most general small s behaviour of the Painlevé V equation in σ-form known from the work of Jimbo. An analogous study is carried out for the the hard edge scaling limit of the random matrix average, which we have previously evaluated in terms of a Painlevé III$'$ transcendent in σ-form. An application of the latter result is given to the rapid evaluation of a Hankel determinant appearing in a recent work of Conrey, Rubinstein and Snaith relating to the derivative of the Riemann zeta function.

1. Introduction

The Laguerre unitary ensemble (LUE$_N$) refers to the eigenvalue probability density function (p.d.f.)

$$\frac{1}{C_{N,a}}\prod_{l=1}^{N}\lambda_l^a e^{-\lambda_l}\prod_{1\le j<k\le N}(\lambda_k-\lambda_j)^2, \tag{1.1}$$

where

$$C_{N,a}:=\prod_{j=0}^{N-1}\Gamma(j+1)\Gamma(j+a+1), \tag{1.2}$$

with support on $\lambda_l\in[0,\infty)$. For $a\in\mathbb{Z}_{\ge 0}$, this eigenvalue p.d.f. is realised by non-negative matrices $X^\dagger X$ where X is an $M\times N$ complex Gaussian matrix and $a=M-N$. In our work [3] the average over (1.1)

$$\tilde{E}_{2,N}((0,s);a,\mu;\xi):=\frac{C_{N,a}}{C_{N,a+\mu}}\Big\langle\prod_{l=1}^{N}(1-\xi\chi_{(0,s)}^{(l)})(\lambda_l-s)^\mu\Big\rangle_{\mathrm{LUE}_N}, \tag{1.3}$$

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where $\chi_J^{(l)} = 1$ for $\lambda_l \in J$, $\chi_J^{(l)} = 0$ otherwise, and the normalisation is chosen so that

$$\tilde{E}_{2,N}((0,s);a,\mu;\xi)\Big|_{s=0} := 1, \tag{1.4}$$

was characterised as a τ-function for the Painlevé V system. As a consequence, it was shown that

$$W_N(s;a,\mu;\xi) := s\frac{d}{ds}\log\left(s^{-N\mu}\tilde{E}_{2,N}((0,s);a,\mu;\xi)\right), \tag{1.5}$$

satisfies the Jimbo-Miwa-Okamoto σ-form of the Painlevé V equation

$$(s\sigma_V'')^2 - \left[\sigma_V - s\sigma_V' + 2(\sigma_V')^2 + (\sum_{j=0}^{3}\nu_j)\sigma_V'\right]^2 + 4\prod_{j=0}^{3}(\nu_j + \sigma_V') = 0, \tag{1.6}$$

with

$$\nu_0 = 0, \quad \nu_1 = -\mu, \quad \nu_2 = N + a, \quad \nu_3 = N, \quad \sum_{j=0}^{3}\nu_j = 2N + a - \mu. \tag{1.7}$$

For this to uniquely characterise W_N, a boundary condition must be specified. However in [3] only in the cases $\mu = 0$ and $\mu = 2$ were a boundary condition specified for general ξ.

Also considered in [3] was the hard edge limiting average

$$\tilde{E}_N^{\rm hard}(t;a,\mu;\xi) := \lim_{N\to\infty}\Big(\frac{C_{N,a}}{C_{N,a+\mu}}\tilde{E}_{2,N}((0,\frac{t}{4N});a,\mu;\xi)\Big). \tag{1.8}$$

It was shown that

$$\tilde{E}_2^{\rm hard}(t;a,\mu;\xi) = \exp\int_0^t u^h(s;a,\mu;\xi)\,\frac{ds}{s}, \tag{1.9}$$

where setting

$$u^h(s;a,\mu;\xi) = -(\sigma_{\rm III'}(s) + \frac{\mu(\mu+a)}{2}), \tag{1.10}$$

the function $\sigma_{\rm III'}(s)$ satisfies the Jimbo-Miwa-Okamoto σ-form of the Painlevé III$'$ equation

$$(s\sigma_{\rm III'}'')^2 - v_1v_2(\sigma_{\rm III'}')^2 + \sigma_{\rm III'}'(4\sigma_{\rm III'}' - 1)(\sigma_{\rm III'} - s\sigma_{\rm III'}') - \frac{1}{4^3}(v_1 - v_2)^2 = 0, \tag{1.11}$$

with parameters

$$v_1 = a + \mu, \qquad v_2 = a - \mu. \tag{1.12}$$

Again only in the cases $\mu = 0$ and $\mu = 2$ were boundary conditions specified for general ξ.

The aim of this work is to specify the boundary conditions relevant to both (1.5) and (1.6) for general values of the parameters. In the case of (1.5) this is done by writing the average (1.3) in its equivalent determinant form and evaluating the matrix elements in terms of certain ${}_1F_1$ hypergeometric functions. With the small s behaviour of the matrix elements determined, it turns out that the determinant is

such that its corresponding small s behaviour can readily be deduced. The small s asymptotic form of (1.3) then follows immediately. Scaling this asymptotic form as required by (1.8) then gives the small t behaviour of $\tilde{E}_N^{\rm hard}(t;a,\mu;\xi)$ and the small s behaviour of $u^h(s;a,\mu;\xi)$.

The general small s asymptotic form of the permitted solutions of (1.6) and (1.11) have been given by Jimbo [5]. As part of this study the boundary conditions found here are compared against these general forms. It is found that in both cases only one of the two branches permitted by the general solution is present in our random matrix problem.

As an application of our results we specify the rapid computation of the power series expansion of a certain Hankel determinant of Bessel functions. The latter is known form our work [3] to be a special case of $\tilde{E}_N^{\rm hard}(t;a,\mu;\xi)$. The coefficients in the power series appear in an asymptotic formula obtained recently by Conrey, Rubinstein and Snaith [1] for the integer moments of the derivative of the characteristic polynomial of a unitary random matrix. This in turn has application to the study of the derivative of the Riemann zeta function on the critical line.

2. Small s Expansion of $\tilde{E}_{2,N}((0,s);a,\mu;\xi)$

A standard result in random matrix theory, which in fact goes back to an identity of Heine (see [7]) expresses the random matrix average (1.3) as a determinant

$$\tilde{E}_{2,N}((0,s);a,\mu;\xi) = \frac{N!C_{N,a}}{C_{N,a+\mu}} \det[w_{j+k}]_{j,k=0,\dots,N-1}, \tag{2.1}$$

where

$$w_n := \int_0^\infty d\lambda\ (1-\xi\chi_{(0,s)})(\lambda-s)^\mu \lambda^{n+a} e^{-\lambda}. \tag{2.2}$$

Unless μ is a non-negative integer (2.2) is not well defined for s real and positive, which is the domain of interest. To remedy this, we note that simple manipulation gives

$$w_n := \int_s^\infty d\lambda\ (\lambda-s)^\mu \lambda^{n+a} e^{-\lambda} + (1-\xi)\int_0^s d\lambda\ (\lambda-s)^\mu \lambda^{n+a} e^{-\lambda}, \tag{2.3}$$

and in the second integral of this expression write $(\lambda-s)^\mu = e^{\mu\log(\lambda-s)}$ with $-\pi < \arg\log(\lambda-s) \le \pi$. We then obtain

$$w_n := \int_s^\infty d\lambda\ (\lambda-s)^\mu \lambda^{n+a} e^{-\lambda} + (1-\xi)e^{\pi i\mu}\int_0^s d\lambda\ (s-\lambda)^\mu \lambda^{n+a} e^{-\lambda}, \tag{2.4}$$

which is well defined for $\Re(\mu) > -1$ and $\Re(a) > -1$ for $s > 0$ with the additional constraint $\Re(\mu+a) > -1$ at $s = 0$.

We seek the leading terms in the small s expansion of (2.4). These can be read off from an explicit evaluation in terms of the ${}_1F_1$ confluent hypergeometric function [8].

Proposition 2.1. *Subject to the conditions $\Re(\mu) > -1$, $\Re(a) > -1$, $\Re(\mu+a) > -1$ and $\mu + a \notin \mathbb{Z}_{\geq 0}$ we have*

$$w_n = a_n(s) + s^{n+\mu+a+1} b_n(s), \tag{2.5}$$

where $a_n(s), b_n(s)$ are analytic about $s = 0$ and given explicitly by

$$\begin{aligned} a_n(s) &= \Gamma(\mu+n+a+1) e^{-s} {}_1F_1(-a-n; -\mu-a-n; s), \\ b_n(s) &= \frac{\Gamma(\mu+1)\Gamma(n+a+1)}{\Gamma(\mu+n+a+2)} \left((1-\xi) e^{\pi i \mu} - \frac{\sin \pi a}{\sin \pi(\mu+a)} \right) \\ &\quad \times e^{-s} {}_1F_1(\mu+1; \mu+a+n+2; s). \end{aligned} \tag{2.6}$$

In particular, under the above conditions,

$$w_n \underset{s\to 0}{\sim} a_n(0) + s a_n'(0) + s^{n+\mu+a+1} b_n(0), \tag{2.7}$$

where

$$\begin{aligned} a_n(0) &= \Gamma(\mu+n+a+1), \\ a_n'(0) &= -\mu \Gamma(\mu+n+a), \\ b_n(0) &= \frac{\Gamma(\mu+1)\Gamma(n+a+1)}{\Gamma(\mu+n+a+2)} \left((1-\xi) e^{\pi i \mu} - \frac{\sin \pi a}{\sin \pi(\mu+a)} \right). \end{aligned} \tag{2.8}$$

Proof. The results (2.7) and (2.8) are immediate corollaries of (2.5) and (2.6) and the fact that

$${}_1F_1(\gamma; \alpha; s) = 1 + \frac{\gamma}{\alpha} s + \mathrm{O}(s^2).$$

To derive (2.5), we note that simple manipulation shows

$$\int_s^\infty d\lambda\, (\lambda - s)^\mu \lambda^{n+a} e^{-\lambda} = s^{a+n} e^{-s} \int_0^\infty d\lambda\, (1 + \lambda/s)^{n+a} \lambda^\mu e^{-\lambda}.$$

But with

$$W_{k,m}(z) = \frac{z^k e^{-z/2}}{\Gamma(1/2 - k + m)} \int_0^\infty dt\, (1 + t/z)^{k-1/2+m} t^{-k-1/2+m} e^{-t},$$

specifying the Whittaker function, it is known that [8]

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(1/2-k-m)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(1/2-k+m)} M_{k,-m}(z)$$

where

$$M_{k,m}(z) = z^{m+1/2} e^{-z/2} {}_1F_1(1/2 - k + m; 2m+1; z).$$

Consequently

$$\begin{aligned} \int_s^\infty d\lambda\, (\lambda - s)^\mu \lambda^{n+a} e^{-\lambda} &= \Gamma(\mu+a+n+1) e^{-s} {}_1F_1(-a-n; -\mu-a-n; s) \\ &+ \frac{\Gamma(\mu+1)\Gamma(-\mu-a-n-1)}{\Gamma(-a-n)} s^{\mu+a+n+1} e^{-s} {}_1F_1(\mu+1; \mu+a+n+2; s). \end{aligned} \tag{2.9}$$

The left-hand side of (2.9) exists for $\Re(\mu) > -1$ if $s > 0$ and $\Re(\mu + a) > -1$ if $s = 0$, whereas the right-hand side is valid in this parameter domain except for $\mu + a + n \in \mathbb{Z}_{\geq 0}$, and in this case the individual terms have a simple pole at

$a+n \notin \mathbb{Z}_{\geq 0}$ or are undefined when $a+n \in \mathbb{Z}_{\geq 0}$. Needless to say the sum of the terms on the right-hand side has the same analytic character as the left-hand side.

Regarding the second integral in (2.4), we first note that a simple change of variables gives

$$\int_0^s d\lambda\,(s-\lambda)^\mu \lambda^{n+a} e^{-\lambda} = s^{n+1+a+\mu} e^{-s} \int_0^1 dx\,(1-x)^{n+a} x^\mu e^{sx}.$$

But the integral on the right hand side is the Euler integral representation of the ${}_1F_1$ function, which shows

$$\begin{aligned}(2.10)\quad \int_0^s & d\lambda\,(s-\lambda)^\mu \lambda^{n+a} e^{-\lambda} \\ &= \frac{\Gamma(\mu+1)\Gamma(a+n+1)}{\Gamma(\mu+a+n+2)} s^{\mu+a+n+1} e^{-s}\,{}_1F_1(\mu+1;\mu+a+n+2;s).\end{aligned}$$

This latter relation is valid for $\Re(\mu) > -1$ and $\Re(a) > -1$ when $s > 0$. Substituting (2.9) and (2.10) in (2.4) and using the appropriate gamma function identities gives (2.5), (2.6). □

When $\mu + a \in \mathbb{Z}_{\geq 0}$ we have to consider two exceptional cases where one of the hypergeometric functions are not defined - the first when $a+n \in \mathbb{Z}_{\geq 0}$ for which the hypergeometric function is indeterminate, and the second when $a+n \notin \mathbb{Z}_{\geq 0}$ and the hypergeometric function has a simple pole. These two cases can be recovered by taking suitable limits and we just state the final results.

Proposition 2.2. *When $\mu + a = j \in \mathbb{Z}_{\geq 0}$ and $a + n = k \in \mathbb{Z}_{\geq 0}$ with $n + j \geq k$ we have*

$$\begin{aligned}(2.11)\quad w_n = k! e^{-s} \Bigg\{ & \sum_{l=0}^{k} \frac{(n+j-l)!}{(k-l)!\,l!} s^l \\ & + (-1)^{n+j+k}(1-\xi)\frac{(n+j-k)!}{(n+j+1)!} s^{n+j+1}\,{}_1F_1(n+j+1-k;n+j+2;s)\Bigg\},\end{aligned}$$

and to leading order in small s we have

(2.12)

$$w_n \underset{s\to 0}{\sim} (n+j)! - (n+j-k)(n+j-1)!s + (-1)^{n+j+k}(1-\xi)\frac{(n+j-k)!k!}{(n+j+1)!} s^{n+j+1}.$$

Note that the condition $n + j \geq k$ is the same as $\mu \geq 0$, which falls within the domain of interest. The key difference of (2.12) with (2.7) and (2.8) is that the non-analytic term is now polynomial and the second part of this term is absent having been cancelled by a counterbalancing term.

Proposition 2.3. *When $\mu + a = j \in \mathbb{Z}_{\geq 0}$ and $a + n \notin \mathbb{Z}_{\geq 0}$ we have*

$$
\begin{aligned}
(2.13)\quad w_n = e^{-s}\Bigg\{ &\sum_{l=0}^{n+j} \frac{(-a-n)_l (n+j-l)!}{l!}(-s)^l \\
&+ \frac{\Gamma(\mu+1)\Gamma(a+n+1)}{(n+j+1)!}(1-\xi)e^{i\pi\mu} s^{n+j+1} {}_1F_1(\mu+1; n+j+2; s) \\
&+ (-1)^j \frac{\sin\pi a}{\pi} \frac{\Gamma(\mu+1)\Gamma(a+n+1)}{(n+j+1)!} s^{n+j+1} \\
&\times \sum_{l=0}^{\infty} [\psi(l+1) + \psi(n+j+l+2) - \psi(\mu+l+1) - \log s] \frac{(\mu+1)_l}{(n+j+2)_l}\frac{s^l}{l!} \Bigg\},
\end{aligned}
$$

and its leading order behaviour for small s is

$$
\begin{aligned}
(2.14)\quad w_n &\underset{s\to 0}{\sim} (n+j)! + (a-j)(n+j-1)! s \\
&+ \frac{(a-j)_{n+j+1}}{(n+j+1)!} s^{n+j+1} \left[\frac{\pi e^{-i\pi a}}{\sin \pi a}(1-\xi) + \psi(1) + \psi(n+j+2) - \psi(\mu+1) - \log s \right].
\end{aligned}
$$

The expansion (2.14) differs significantly from (2.7) and (2.8) because of the presence of logarithmic terms which now replace the non-analytic contributions of the generic case.

Corollary 2.1. *Under generic conditions on $\mu + a$ we have*

$$
\begin{aligned}
(2.15)\quad &\det[w_{j+k}]_{j,k=0,\dots,N-1} \\
&= \det[\Gamma(\mu+a+1+j+k)]_{j,k=0,\dots,N-1} \\
&- \mu s \det[\Gamma(\mu+a+j) \ \ \Gamma(\mu+a+1+j+k)]_{\substack{j=0,\dots,N-1 \\ k=1,\dots,N-1}} + \mathrm{O}(s^2) \\
&+ s^{\mu+a+1} b_0(0) \det[\Gamma(\mu+a+3+j+k)]_{j,k=0,\dots,N-2} \{1 + \mathrm{O}(s)\} \\
&+ \mathrm{O}(s^{2(\mu+a+1)}).
\end{aligned}
$$

Proof. According to (2.7)

$$
\begin{aligned}
&\det[w_{j+k}]_{j,k=0,\dots,N-1} \\
&\underset{s\to 0}{\sim} \det[a_{j+k}(0) + s a'_{j+k}(0) + s^{\mu+a+1+j+k} b_{j+k}(0)]_{j,k=0,\dots,N-1} \\
&\underset{s\to 0}{\sim} \det[a_{j+k}(0)]_{j,k=0,\dots,N-1} + s[s]\det[a_{j+k}(0) + s a'_{j+k}(0)]_{j,k=0,\dots,N-1} \\
&\qquad + s^{\mu+a+1} b_0(0) \det[a_{j+k+2}(0)]_{j,k=0,\dots,N-2},
\end{aligned}
$$

where $[s]P(s)$ denotes the coefficient of s in $P(s)$. Recalling the explicit formula for $a_n(0)$ as given in (2.8) we obtain the constant term and the term proportional to $s^{\mu+a+1}$ in (2.15). It remains to compute the coefficient of s, which according to (2.8) has the explicit form

$$
(2.16)\qquad [s]\det[\Gamma(\mu+a+1+j+k) - \mu s \Gamma(\mu+a+j+k)]_{j,k=0,\dots,N-1}.
$$

Using the linearity formula

$$\det[\mathbf{a}_1 \cdots \mathbf{a}_j + \mathbf{b}_j \cdots \mathbf{a}_n] = \det[\mathbf{a}_1 \cdots \mathbf{a}_j \cdots \mathbf{a}_n] + \det[\mathbf{a}_1 \cdots \mathbf{b}_j \cdots \mathbf{a}_n],$$

where the **a**'s and **b**'s are column vectors, on each column of the determinant we see that of the terms proportional to s only the one obtained from expanding the first column in non-zero (all the rest result in two identical columns), and the determinant given by (2.15) results. □

It remains to evaluate the determinants. For this task we make use of the identity [6]

$$\det[\Gamma(z_k + j)]_{j,k=0,\dots,n-1} = \prod_{j=0}^{n-1} \Gamma(z_j) \prod_{0 \le j < k \le n-1} (z_k - z_j).$$

After straightforward manipulations, gamma function evaluations of all the determinants in (2.15) can be obtained. Substituting in (2.1), and recalling that the normalisation is such that at $s = 0$ $\tilde{E}_N$ is equal to unity, we obtain the sought small s expansion of $\tilde{E}_N$ and thus W_N valid for general values of the parameters.

Proposition 2.4. *For $\Re(\mu) > -1$, $\Re(a) > -1$ and $\mu + a \notin \mathbb{Z}_{\ge 0}$ we have*

$$\begin{aligned}(2.17)\quad \tilde{E}_{2,N}((0,s);a,\mu;\xi) &= 1 - \frac{\mu N}{\mu + a} s + \mathrm{O}(s^2) \\ &+ \frac{\Gamma(\mu+1)\Gamma(a+1)\Gamma(\mu+a+N+1)}{\Gamma^2(\mu+a+2)\Gamma(\mu+a+1)\Gamma(N)} \left((1-\xi)e^{\pi i \mu} - \frac{\sin \pi a}{\sin \pi(\mu+a)} \right) s^{\mu+a+1} \left\{ 1 + \mathrm{O}(s) \right\} \\ &+ \mathrm{O}(s^{2(\mu+a+1)}),\end{aligned}$$

and consequently

$$\begin{aligned}(2.18)\quad W_N(s;a,\mu;\xi) &= -N\mu - \frac{\mu N}{\mu + a} s + \mathrm{O}(s^2) \\ &+ \frac{\Gamma(\mu+1)\Gamma(a+1)\Gamma(\mu+a+N+1)}{\Gamma(\mu+a+2)\Gamma^2(\mu+a+1)\Gamma(N)} \left((1-\xi)e^{\pi i \mu} - \frac{\sin \pi a}{\sin \pi(\mu+a)} \right) s^{\mu+a+1} \left\{ 1 + \mathrm{O}(s) \right\} \\ &+ \mathrm{O}(s^{2(\mu+a+1)}).\end{aligned}$$

In the first exceptional case $\mu + a = j \in \mathbb{Z}_{\ge 0}$ and $a = k \in \mathbb{Z}_{\ge 0}$ with $j \ge k$ one can still use (2.17) but omitting the term involving the ratio of sines, in the case $j = 0$, or the whole term if $j > 0$. The situation of the other exceptional case $\mu + a = j \in \mathbb{Z}_{\ge 0}$ and $a \notin \mathbb{Z}_{\ge 0}$ is more complicated and more so for larger j, and we only give the examples of $j = 0, 1$.

Proposition 2.5. *For $\Re(\mu) > -1$, $\Re(a) > -1$ with $\mu + a = 0$ we have*

$$\begin{aligned}(2.19)\quad \tilde{E}_{2,N}((0,s);a,\mu=-a;\xi) &= 1 \\ &+ \left\{ -1 + \frac{\pi a}{\sin \pi a} e^{-i\pi a}(1-\xi) + a\left[2\psi(2) + \psi(1) - \psi(1-a) - \psi(N+1) - \log s\right] \right\} Ns \\ &+ \mathrm{o}(s).\end{aligned}$$

For $\mu + a = 1$ we have

$$
\begin{aligned}
(2.20)\quad & \tilde{E}_{2,N}((0,s);a,\mu=1-a;\xi) = 1+(a-1)Ns \\
&+\frac{a(a-1)}{4}\Big\{\frac{\pi}{\sin\pi a}e^{-i\pi a}(1-\xi)+2\psi(3)+\psi(2)-\psi(2-a)-\psi(N+2)-\log s\Big\}(N+1)Ns^2 \\
&+\mathrm{o}(s^2).
\end{aligned}
$$

3. Comparison with the Jimbo solution

The small s expansion of the most general solution permitted by (1.6), or more precisely its corresponding τ-function (see (3.2) below) has been determined by Jimbo [5]. However in [5] the equation (1.6) is not treated directly. Instead the discussion is based on the equation

$$
\begin{aligned}
(3.1)\quad & (s\zeta'')^2 - \big[\zeta - s\zeta' + 2(\zeta')^2 - (2\theta_0+\theta_\infty)\zeta'\big]^2 \\
&\quad + 4\zeta'(\zeta'-\theta_0)(\zeta'-\frac{1}{2}(\theta_0-\theta_s+\theta_\infty))(\zeta'-\frac{1}{2}(\theta_0+\theta_s+\theta_\infty)) = 0,
\end{aligned}
$$

and the small s behaviour of the corresponding τ-function $\tau_V(s)$, specified by the the requirement that

$$
(3.2)\qquad \zeta(s) = s\frac{d}{ds}\log\tau_V(s) + \frac{1}{2}(\theta_0+\theta_\infty)s + \frac{1}{4}[(\theta_0+\theta_\infty)^2-\theta_s^2],
$$

was determined.

Comparison of (3.1), (3.2) with (1.6), (1.5) shows that for the parameters (1.7)

$$
(3.3)\qquad \tilde{E}_{2,N}((0,s);a,\mu;\xi) = s^{N^2+N(a+\mu)}e^{-(N+a/2)s}\tau_V(s),
$$

while in general

$$
(3.4)\qquad \theta_0 = -\nu_1,\quad \theta_s = \nu_2-\nu_3,\quad \theta_\infty = \nu_1-\nu_2-\nu_3.
$$

Note that for the parameters (1.7) we thus thus have

$$
(3.5)\qquad \theta_0 = \mu,\quad \theta_s = a,\quad \theta_\infty = -2N-a-\mu.
$$

The relevant result from [5] can now be presented. It states that the most general small s behaviour of $\tau_V(s)$ permitted by the equation (3.1) is

$$
\begin{aligned}
\tau_V(s) = Cs^{(\sigma^2-\theta_\infty^2)/4}\Bigg\{ 1 - \frac{\theta_\infty(\theta_s^2-\theta_0^2+\sigma^2)}{4\sigma^2}s \\
+ u\frac{\Gamma^2(-\sigma)}{\Gamma^2(2+\sigma)}\frac{\Gamma(1+\frac{\theta_s+\theta_0+\sigma}{2})\Gamma(1+\frac{\theta_s-\theta_0+\sigma}{2})\Gamma(1+\frac{\theta_\infty+\sigma}{2})}{\Gamma(\frac{\theta_s+\theta_0-\sigma}{2})\Gamma(\frac{\theta_s-\theta_0-\sigma}{2})\Gamma(\frac{\theta_\infty-\sigma}{2})}s^{1+\sigma} \\
+ \frac{1}{u}\frac{\Gamma^2(\sigma)}{\Gamma^2(2-\sigma)}\frac{\Gamma(1+\frac{\theta_s+\theta_0-\sigma}{2})\Gamma(1+\frac{\theta_s-\theta_0-\sigma}{2})\Gamma(1+\frac{\theta_\infty-\sigma}{2})}{\Gamma(\frac{\theta_s+\theta_0+\sigma}{2})\Gamma(\frac{\theta_s-\theta_0+\sigma}{2})\Gamma(\frac{\theta_\infty+\sigma}{2})}s^{1-\sigma} \\
+ \mathrm{O}(|s|^{2(1-\Re(\sigma))})\Bigg\},
\end{aligned}
\tag{3.6}
$$

where C is a normalisation constant, while u and σ are arbitrary parameters. The above result was derived subject to the conditions $\theta_0, \theta_s \notin \mathbb{Z}$, $\frac{1}{2}(\theta_\infty \pm \sigma) \notin \mathbb{Z}$, $\frac{1}{2}(\theta_s \pm \theta_0 \pm \sigma) \notin \mathbb{Z}$ and that $0 < \Re(\sigma) < 1$ (a distinct solution was presented for $\sigma = 0$). These conditions therefore strictly apply only to the generic or transcendental solutions of the fifth Painlevé equation. For generic parameter values the terms given in (3.6) uniquely specify all the subsequent terms in the convergent Puisuex-type expansion for $\zeta(s)$ about $s = 0$

$$
\zeta(s) = \sum_{j=0}^{\infty}\sum_{|k|\le j} c_{j,k}s^{j+k\sigma}, \tag{3.7}
$$

i.e. with any two of $c_{1,0}, c_{1,1}$ or $c_{1,-1}$ given.

To relate this to $\tilde{E}_{2,N}$, we see from (3.3) and (3.5) that we require $\sigma^2 = (a+\mu)^2$ and thus we can choose

$$
\sigma = a + \mu. \tag{3.8}
$$

This relation, $\sigma = \theta_0 + \theta_s$, is a violation of one of the strict conditions given above and is in fact a sufficient condition for a classical solution, along with the necessary condition $\theta_0 + \theta_s + \theta_\infty = -2N \in \mathbb{Z}$, which is the type of solution that we are dealing with here. However we conjecture that Jimbo's conditions can be relaxed to accommodate such solutions and the corresponding formulae (or limiting forms if necessary) still hold. With this choice of σ the coefficient of $s^{1-\sigma}$ in (3.6) contains a factor of

$$
\frac{1}{\Gamma(\frac{\theta_\infty+\sigma}{2})} = \frac{1}{\Gamma(-N)}
$$

and thus vanishes. Simplifying the other terms gives

$$\tau_V(s) \sim Cs^{-N^2-N(a+\mu)}\Bigg\{1+\frac{(2N+a+\mu)a}{2(a+\mu)}s + u\frac{\sin\pi\mu}{\sin\pi(a+\mu)}\frac{\Gamma(a+1)\Gamma(\mu+1)\Gamma(N+1+a+\mu)}{\Gamma^2(2+a+\mu)\Gamma(1+a+\mu)\Gamma(N)}s^{1+a+\mu}\Bigg\}.$$

Substituting in (3.3) we see that this is in precise agreement with (2.17) provided we choose

$$u\frac{\sin\pi\mu}{\sin\pi(a+\mu)} = (1-\xi)e^{\pi i\mu} - \frac{\sin\pi a}{\sin\pi(a+\mu)} \tag{3.9}$$

4. The hard edge limit

The hard edge limit is defined by (1.8). However, only in the cases $\mu = 0$, $\mu = 2$ do we know how to prove its existence for general ξ (in the case $\mu = 0$ $\tilde{E}_{2,N}$ can be written as a Fredholm determinant, while the case $\mu = 2$ is related to this via differentiation). However a log-gas viewpoint ([2]) indicates that the limit will be well defined, and moreover we expect that it can be taken term-by-term in the small s expansion of $\tilde{E}_{2,N}$. In this section we will show that taking the hard edge limit of the small s expansion (2.17) give rise to an initial condition for the solution of (1.11) consistent with that allowed by Jimbo's theory of the small s expansion of the Painlevé III$'$ equation. From a practical perspective this specifies $\tilde{E}_2^{\rm hard}$ for general values of the parameters according to (1.9), while from a theoretical viewpoint it lends weight to the belief that (1.9) is indeed the correct limiting evaluation for general values of the parameters.

Under the assumption that the hard edge limit can be taken term-by-term in the small s expansion of Proposition (2.4) is immediate.

Corollary 4.1. *For $\Re(\mu) > -1$, $\Re(a) > -1$ and $\mu + a \notin \mathbb{Z}_{\geq 0}$ we have*

$$\tilde{E}_2^{\rm hard}(s;a,\mu;\xi) = 1 - \frac{\mu}{4(a+\mu)}s + {\rm O}(s^2) + \frac{\Gamma(\mu+1)\Gamma(a+1)}{\Gamma^2(\mu+a+2)\Gamma(\mu+a+1)}\left((1-\xi)e^{\pi i\mu} - \frac{\sin\pi a}{\sin\pi(\mu+a)}\right)\left(\frac{s}{4}\right)^{\mu+a+1}\{1+{\rm O}(s)\} + {\rm O}(s^{2(\mu+a+1)}), \tag{4.1}$$

and consequently the σ-function $\sigma_{\rm III'}(s)$ in (1.10) has the small s expansion

$$\sigma_{\rm III'}(s) = -\frac{\mu(\mu+a)}{2} + \frac{\mu}{4(\mu+a)}s + {\rm O}(s^2) - \frac{\Gamma(\mu+1)\Gamma(a+1)}{\Gamma(\mu+a+2)\Gamma^2(\mu+a+1)}\left((1-\xi)e^{\pi i\mu} - \frac{\sin\pi a}{\sin\pi(\mu+a)}\right)\left(\frac{s}{4}\right)^{\mu+a+1}\{1+{\rm O}(s)\} + {\rm O}(s^{2(\mu+a+1)}). \tag{4.2}$$

Some examples of exceptional cases not covered by the preceding corollary are the following. They are obtained by taking the hard edge limit of (2.19) and (2.20).

Corollary 4.2. *For $\Re(\mu) > -1$, $\Re(a) > -1$ and $\mu + a = 0$ we have*

$$\tag{4.3} \begin{aligned} \tilde{E}_2^{\mathrm{hard}}(s; a, \mu = -a; \xi) = 1 \\ + \Big\{ -1 + \frac{\pi a}{\sin \pi a} e^{-\pi i a}(1-\xi) + a[2\psi(2) + \psi(1) - \psi(1-a) - \log(s/4)] \Big\} \frac{s}{4} \\ + \mathrm{o}(s), \end{aligned}$$

whilst for $\mu + a = 1$ we have

$$\tag{4.4} \begin{aligned} \tilde{E}_2^{\mathrm{hard}}(s; a, \mu = 1-a; \xi) = 1 + (a-1)\frac{s}{4} \\ + \frac{a(a-1)}{4} \Big\{ \frac{\pi}{\sin \pi a} e^{-\pi i a}(1-\xi) + 2\psi(3) + \psi(2) - \psi(2-a) - \log(s/4) \Big\} \Big(\frac{s}{4}\Big)^2 \\ + \mathrm{o}(s^2). \end{aligned}$$

To compare these results to the small independent variable expansions given by Jimbo in the theory of III′ , we must first undertake some preliminary calculations as the equation (1.11) is not directly studied in [5]. Rather the equation studied is

$$\tag{4.5} (t\zeta'')^2 = 4\zeta'(\zeta'-1)(\zeta - t\zeta') + \Big(\frac{v_1 + v_2}{2} - v_1 \zeta' \Big)^2,$$

where we have identified $\theta_0 = -v_2$, $\theta_\infty = -v_1$ (θ_0, θ_∞ are the parameters appearing in [5]). In terms of $\zeta(t)$ the τ-function $\tau_{\mathrm{III}'}(t)$ is specified by the requirement that

$$\tag{4.6} \zeta(t) = t \frac{d}{dt} \log \tau_{\mathrm{III}'}(t) + \frac{v_2^2 - v_1^2}{4} + t,$$

and it is the small t expansion of $\tau_{\mathrm{III}'}(t)$ presented in [5]. Comparison of (4.5) and (1.11) shows that

$$\tag{4.7} \zeta(t) = -\sigma_{\mathrm{III}'}(s) + \frac{v_1(v_2 - v_1)}{4} + \frac{s}{4}, \qquad t = \frac{s}{4}.$$

Recalling (1.10), (1.9), (4.7) and (4.6) we see

$$\tag{4.8} \tilde{E}_2^{\mathrm{hard}}(s; a, \mu; \xi) = t^{(v_2^2 - v_1^2)/4} \tau_{\mathrm{III}'}(t).$$

In [5] the most general small t expansion of $\tau_{\mathrm{III}'}(t)$ as permitted by (4.5) is presented. It reads

$$
\begin{aligned}
(4.9)\quad \tau_{\mathrm{III}'}(t) = Ct^{(\sigma^2-v_2^2)/4}\Bigg\{ 1 + \frac{v_1v_2-\sigma^2}{2\sigma^2}t \\
- u\frac{\Gamma^2(-\sigma)}{\Gamma^2(2+\sigma)}\frac{\Gamma(1+\frac{v_2+\sigma}{2})\Gamma(1+\frac{v_1+\sigma}{2})}{\Gamma(\frac{v_2-\sigma}{2})\Gamma(\frac{v_1-\sigma}{2})}t^{1+\sigma} \\
- \frac{1}{u}\frac{\Gamma^2(\sigma)}{\Gamma^2(2-\sigma)}\frac{\Gamma(1+\frac{v_2-\sigma}{2})\Gamma(1+\frac{v_1-\sigma}{2})}{\Gamma(\frac{v_2+\sigma}{2})\Gamma(\frac{v_1+\sigma}{2})}t^{1-\sigma} \\
+ \mathrm{O}(|t|^{2(1-\Re(\sigma))})\Bigg\},
\end{aligned}
$$

where as in (3.6) C is a normalisation, while u and σ are arbitrary parameters. This result was established under the assumptions that $\frac{1}{2}(v_1 \pm \sigma) \notin \mathbb{Z}$ and $\frac{1}{2}(v_2 \pm \sigma) \notin \mathbb{Z}$ along with $0 < \Re\sigma < 1$ (for $\sigma = 0$ a distinct solution is given).

To see that this structure is consistent with (4.1) and (4.8), recalling (1.12) we see that for the right hand side of (4.8) to tend to 1 as t tends to zero we must have $C = 1$ and $\sigma = \pm v_1$. Again this is a violation of first condition given above but we conjecture that the formulae have meaning under the following limiting procedure and are correct. Choosing the positive sign for definiteness, and then writing

$$
\frac{u}{\Gamma(\frac{v_1-\sigma}{2})} = \frac{u(v_1-\sigma)}{2\Gamma(1+\frac{v_1-\sigma}{2})}
$$

we see that requiring

$$
\frac{u}{2}(v_1-\sigma) \to \tilde{u}\frac{\sin\pi v_1}{\pi} \quad \text{as} \quad \sigma \to v_1,
$$

(4.9) reads

$$
\begin{aligned}
(4.10)\quad \tau_{\mathrm{III}'}(t) \sim t^{(v_1^2-v_2^2)/4}\Bigg\{ 1 + \frac{v_1v_2-v_1^2}{2v_1^2}t \\
+ \tilde{u}\frac{\sin\pi(v_1-v_2)/2}{\sin\pi v_1}\frac{\Gamma(1-\frac{v_2-v_1}{2})\Gamma(1+\frac{v_2+v_1}{2})}{\Gamma^2(2+v_1)\Gamma(1+v_1)}\left(\frac{t}{4}\right)^{1+v_1}\Bigg\}.
\end{aligned}
$$

Recalling again (1.12) and (4.8) we see that this agrees with (4.1) provided

$$
(4.11)\qquad \tilde{u}\frac{\sin\pi\mu}{\sin\pi(a+\mu)} = (1-\xi)e^{\pi i\mu} - \frac{\sin\pi a}{\sin\pi(a+\mu)},
$$

(cf. (3.9)).

5. Application

In a recent work relating to the application of random matrix theory to the study of moments of the derivative of the Riemann zeta-function, Conrey, Rubinstein and Snaith [1] obtained two asymptotic expressions associated with the derivative of characteristic polynomials for random unitary matrices. With U a Haar distributed element of the unitary group $U(N)$, and $e^{i\theta_1},\dots,e^{i\theta_N}$ its eigenvalues, let

$$\Lambda_A(s) = \prod_{j=1}^{N}(1 - s e^{-i\theta_j}), \tag{5.1}$$

and

$$\mathcal{Z}_A(s) = e^{-\pi i N/2} e^{i\sum_{n=1}^{N}\theta_n/2} s^{-N/2}\Lambda_A(s), \tag{5.2}$$

(note that $\mathcal{Z}_A(e^{i\theta})$ is real for θ real). In terms of this notation, the two results from [1] are

$$\langle |\Lambda_A'(1)|^{2k}\rangle_{A\in U(N)} \underset{N\to\infty}{\sim} b_k N^{k^2+2k}, \tag{5.3}$$

where

$$b_k = (-1)^{k(k+1)/2}\sum_{h=0}^{k}\binom{k}{h}(k+h)! \times [x^{k+h}]\left(e^{-x}x^{-k^2/2}\det[I_{\alpha+\beta-1}(2\sqrt{x})]_{\alpha,\beta=1,\dots,k}\right), \tag{5.4}$$

and

$$\langle |\mathcal{Z}_A'(1)|^{2k}\rangle_{A\in U(N)} \underset{N\to\infty}{\sim} b_k' N^{k^2+2k}, \tag{5.5}$$

where

$$b_k' = (-1)^{k(k+1)/2}(2k)![x^{2k}]\left(e^{-x/2}x^{-k^2/2}\det[I_{\alpha+\beta-1}(2\sqrt{x})]_{\alpha,\beta=1,\dots,k}\right). \tag{5.6}$$

In (5.4) and (5.6) the notation $[x^p]f(x)$ denotes the coefficient of x^p in $f(x)$.

The relevance of these formulae to the present study is that the determinant therein can be identified in terms of $\tilde{E}_2^{\rm hard}$. Thus, we have shown in a previous study [3] that for $a \in \mathbb{Z}_{\ge 0}$

$$\tilde{E}_2^{\rm hard}(s;a,\mu;\xi=1) = A(a,\mu)\left(\frac{2}{\sqrt{s}}\right)^{a\mu} e^{-s/4}\det[I_{\mu+\alpha-\beta}(\sqrt{s})]_{\alpha,\beta=1,\dots,a}. \tag{5.7}$$

where

$$A(a,\mu) = a!\prod_{j=1}^{a}\frac{(j+\mu-1)!}{j!}. \tag{5.8}$$

Interchanging row β by row $a-\beta+1$ ($\beta = 1,\dots,a$ in order) we see from this that

$$
\begin{aligned}
b_k &= \frac{(-1)^k}{A(k,k)} \sum_{h=0}^{k} \binom{k}{h} (k+h)! [x^{k+h}] \tilde{E}_2^{\rm hard}(4x;k,k;\xi=1) \\
b'_k &= \frac{(-1)^k}{A(k,k)} (2k)! [x^{2k}] \left(e^{x/2} \tilde{E}_2^{\rm hard}(4x;k,k;\xi=1) \right)
\end{aligned}
\tag{5.9}
$$

Note that the Painlevé III′ parameters appearing in this solution are $\mu = a = k \in \mathbb{N}$ and $\mu + a = 2k \in 2\mathbb{N}$ and thus we are dealing with the exceptional case of indeterminacy referred to in Section 2. However as was noted there the generic formulae still hold with to the modifications discussed and in particular the σ-function has a small argument expansion of a purely analytic form.

From ([4]) it is known that the determinants in (5.4) and (5.6) can also be expressed as a particular generalised hypergeometric function. Such an observation implies, for instance, that

$$
x^{-k^2/2} \det[I_{\alpha+\beta-1}(2\sqrt{x})]_{\alpha,\beta=1,\dots,k} = \prod_{j=1}^{k} \frac{j!}{\Gamma(j+k)} {}_0F_1^{(1)}(;2k;x_1,\dots,x_k)|_{x_j=x},
\tag{5.10}
$$

where ${}_0F_1^{(1)}(;c;x_1,\dots,x_k)$ has a series development about $x_1,\dots,x_k=0$ with an explicitly given coefficient for an arbitrary term. However this is not a practical or efficient way to compute the coefficients required in (5.4) or (5.6) for moderate or large k as it involves the hook lengths of Young diagrams associated with the partitions of k.

According to (1.9), (1.11) and (4.2)

$$
\tilde{E}_2^{\rm hard}(4x;k,k;\xi=1) = \exp\left(-\int_0^{4x} \frac{ds}{s} \left(\sigma_{\rm III'}(s) + k^2\right) \right),
\tag{5.11}
$$

where $\sigma_{\rm III'}(s)$ satisfies the particular σ-Painlevé III′ equation

$$
(s\sigma''_{\rm III'})^2 + \sigma'_{\rm III'}(4\sigma'_{\rm III'} - 1)(\sigma_{\rm III'} - s\sigma'_{\rm III'}) - \frac{k^2}{16} = 0,
\tag{5.12}
$$

subject to the boundary condition

$$
\sigma_{\rm III'}(s) \underset{s\to 0}{\sim} -k^2 + \frac{s}{8} + \mathrm{O}(s^2), \quad k \in \mathbb{N}.
\tag{5.13}
$$

Substituting

$$
\sigma_{\rm III'}(s) = \eta(s) + \frac{s}{8},
\tag{5.14}
$$

(5.12) reads

$$
(s\eta'')^2 + 4((\eta')^2 - \frac{1}{64})(\eta - s\eta') - \frac{k^2}{4^2} = 0.
\tag{5.15}
$$

We see immediately that $\eta(s)$ can be expanded in an even function of s about $s=0$,

$$
\eta(s) = \sum_{n=0}^{\infty} c_n s^{2n}, \qquad c_0 = -k^2, \quad k \in \mathbb{N}.
\tag{5.16}
$$

Moreover the coefficients can be computed by a recurrence relation.

Proposition 5.1. *Substituting (5.16) in (5.15) shows*

$$c_1 = \frac{1}{64(4k^2-1)}, \tag{5.17}$$

while for $p \geq 2$

$$\begin{aligned} c_p = \frac{1}{2c_1p(2p-1) + (2p-1)/64 - 8pk^2c_1} \\ \times \Bigg(4k^2 \sum_{l=1}^{p-2} (l+1)(p-l)c_{l+1}c_{p-l} \\ - \sum_{l=1}^{p-2} (l+1)(p-l)(2l+1)(2p-2l+1)c_{l+1}c_{p-l} \\ - \sum_{l=1}^{p-1} (1-2l)c_l A_{p-l-1} \Bigg), \end{aligned} \tag{5.18}$$

where

$$A_q = \sum_{l=0}^{q} (l+1)(q-l+1)c_{l+1}c_{q-l+1}. \tag{5.19}$$

Proof. With $h_l := (l+1)(2l+1)c_{l+1}$ we see

$$(s\eta'')^2 = 4\sum_{p=1}^{\infty} H_{p-1}s^{2p}, \qquad H_p = \sum_{l=0}^{p} h_l h_{p-l}, \tag{5.20}$$

and similarly with $a_l := (l+1)c_{l+1}$ we have

$$(s\eta')^2 = 4s^2 \sum_{p=0}^{\infty} A_p s^{2p}, \qquad A_p = \sum_{l=0}^{p} a_l a_{p-l}.$$

It follows from this latter result that

$$\left((\eta')^2 - \frac{1}{64} \right) (\eta - s\eta') = \sum_{p=0}^{\infty} G_p s^{2p}, \tag{5.21}$$

where

$$G_p = \sum_{l=0}^{p} (1-2l)c_l b_{p-l}, \quad b_0 = -\frac{1}{64}, \quad b_p = 4A_{p-1} \quad (p \geq 1).$$

Substituting (5.20) and (5.21) in (5.15) and equating like coefficients of s^{2p} to zero shows that for $p \geq 1$

$$H_{p-1} + G_p = 0.$$

This for $p = 1$ implies (5.17), and for $p > 1$ implies (5.18). □

Using Proposition 5.1 it is straightforward to calculate, via computer algebra, the first k coefficients in (5.16) for any particular value of k. Furthermore use of computer algebra gives the power series up to x^{2k} of

$$\tilde{E}_2^{\rm hard}(4x;k,k;\xi=1) \quad \text{and} \quad e^{x/2}\tilde{E}_2^{\rm hard}(4x;k,k;\xi=1),$$

according to (5.11). From these power series the formulae (5.9) are used to compute b_k and b'_k. In [1] the first 15 values of both b_k and b'_k were tabulated. This can be rapidly extended using the present method. However the resulting rational numbers quickly become unwieldy to record. Let us then be content by presenting just the 16th member of the sequences,

$$b_{16} = \frac{307\cdot 23581\cdot 92867\cdot 760550281759}{2^{272}\cdot 3^{130}\cdot 5^{66}\cdot 7^{42}\cdot 11^{24}\cdot 13^{21}\cdot 17^{16}\cdot 19^{14}\cdot 23^{10}\cdot 29^{6}\cdot 31^{5}\cdot 37^{3}\cdot 41^{2}\cdot 43^{2}\cdot 47\cdot 53\cdot 59\cdot 61},$$

$$b'_{16} = \frac{4148297603\cdot 76230778088705861517484553692172135066713345305 97}{2^{264}\cdot 3^{133}\cdot 5^{66}\cdot 7^{42}\cdot 11^{25}\cdot 13^{21}\cdot 17^{16}\cdot 19^{14}\cdot 23^{11}\cdot 29^{7}\cdot 31^{6}\cdot 37^{3}\cdot 41^{2}\cdot 43^{2}\cdot 47\cdot 53\cdot 59\cdot 61}.$$

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